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On energy arguments applied to the hydrodynamic impact force

R. COINTE, E. FONTAINE[1], B. MOLIN[2] and Y.M. SCOLAN[2]
Ecole Polytechnique, Palaiseau, France (e-mail: raymond.cointe@environment.gouv.fr); [1]*Institut Français du Pétrole, Rueil Malmaison, France (e-mail: emmanuel. fontaine@ifp.fr);* [2]*Ecole Supérieure d'Ingénieurs de Marseille, Marseille, France (e-mail: molin@esim.fr; scolan@esim.fr)*

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Abstract. The hydrodynamic impact problem is investigated within the framework of potential-flow theory. The vertical load acting on the rigid body is derived based on either momentum or energy conservation, and using the concept of added mass together with a homogeneous Dirichlet condition for the potential on the free surface as usually done to model an impact problem. It is demonstrated that the use of this simplified dynamic free-surface condition, instead of the fully nonlinear one, has a direct influence on the computation of the loads. In particular, the equivalence of momentum and energy analysis is in general not recovered. The situation is then highlighted by performing an asymptotic analysis of the two-dimensional blunt-body asymmetric impact problem. The asymptotic solution is given explicitly and validated through comparisons with experimental results. The energy distribution is then studied. It is shown that the contradiction between momentum and energy analysis can be removed, provided that the flux of energy through the jets is taken into account in the energy balance. If the simplified free-surface condition is indeed valid in the far-field, nonlinear terms must be retained near the body, in the spray-root domains. To leading order, the energy distribution during the gravity-free inertia stage does not depend on the blunt-body shape. The general analysis based on momentum or energy conservation suggests that this result also applies for arbitrary body shape as soon as a homogeneous Dirichlet condition can be applied as a dynamical free-surface boundary condition. In this case, and for a constant vertical impact velocity, half the work performed by the body would seem to be transferred to the fluid as kinetic energy within the spray.

Key words: asymmetry, energy distribution, water impact

1. Introduction

The water-entry problem has recently received considerable attention, in particular because of its applications to ship building. Slamming loads with high pressure occur during impact between a blunt body and the free surface of an incompressible fluid. Within the framework of incompressible potential-flow theory, these loads are usually computed using the concept of added mass in a momentum analysis [1–5].

For the water-entry problem of a wedge, Pierson [6] noted an anomaly when the concept of added mass is employed in an energy analysis, because the two approaches yielded different results. When the speed of the body is constant, the results differ by a factor of 2. Payne [7] made the conjecture that the 'difference' might be due to the energy dissipated in the spray sheet. Indeed, the equal division of energy into the spray and the remainder of the fluid was predicted for high-speed prismatic hulls [8]. The problem has been further highlighted by Molin, Cointe and Fontaine [9]. The aim of this paper is to show that there is no contradiction between momentum and energy balances, provided that the flux of energy going into the spray is taken into account in the energy analysis.

A literature review concerning energy loss associated with a sudden motion of a floating body was recently given by Korobkin and Peregrine [10]. For a compressible fluid, the 'lost' energy is to be found in the shock waves which escape from the body. For a sphere that is impulsively started at a constant velocity, half the work done to move the body goes into kinetic energy in the surrounding flow while the other half is carried away with the compression wave [11]. The 'lost' energy may also be related to free-surface effects. In a very interesting study, Korobkin [12] shows how the method of matched asymptotic expansions can be used to describe the evolution of the flow during the first instant of impact between a blunt body and the free surface of a compressible fluid. This acoustic impact theory shows that for large time, when the compression wave is far from the body, the kinetic energy of the spray jet tends to the value of the kinetic energy of the main flow, their sum being equal to the total work done by the body. A similar result is obtained in the present study, where the fluid is assumed to be incompressible. The inertia stage considered here is obtained as the limit of the subsonic regime, as the Mach number M goes to zero.

In an unbounded and incompressible fluid, it is well known that the force acting on a rigid body is equal to its added mass multiplied by its acceleration. Similarly, the rate of change of the energy within the fluid is equal to the work performed by the moving body. In the first part of the present study, hydrodynamic loads for a rigid body moving into an incompressible fluid limited by solid boundaries are derived using both a momentum and an energy analysis. In this configuration, the two methods give the same result for the force. The geometry is then extended to account for the presence of a free surface. The resulting problem is made more complex due to the nonlinearity of the free-surface conditions. In the hydrodynamic impact problem, the added-mass coefficient is classically computed using a homogeneous Dirichlet condition applied at the undisturbed position of the free surface, usually including the so-called wetted correction. It is demonstrated that the use of this simplified free-surface condition has a direct effect on the computed loads and the equivalence between the momentum and the energy analysis is in general not recovered.

In the second part, the situation is further highlighted using an asymptotic analysis. Since the pioneering work of Wagner [8], numerous asymptotic studies have been performed on the problem of a blunt body striking the free surface of an incompressible fluid. For this problem, the perturbation parameter is the ratio between the immersion and the wetted length of the body. A complete survey of the literature on the subject is outside the scope of this paper but excellent reviews of the subject were given by Korobkin and Puchnachov [13], Mitzoguchi and Tanizawa [14] and Faltinsen [15]. In the present study, the classical two-dimensional asymptotic solution ([8], [16–19]) is extended to the case of an asymmetric blunt-body shape. For an arbitrary blunt-body shape, the so-called wetted length is obtained using the displacement potential, as proposed by Korobkin [20]. Earlier results by Toyama [21] are recovered and generalized. The case of an asymmetric blunt-body shape has also been studied recently by Iafrati [22].

The composite solution is one of the simplest examples of an analytical description of a flow with a free surface experiencing violent deformations. As a result, the energy distribution can be studied analytically. Such an analysis was performed recently by Korobkin and Peregrine [10] for a half-submerged sphere moving into a compressible fluid. In the present case, attention is focused on the energy loss due to free-surface effects. By use of the asymptotic solution, it is shown that the flux of momentum going into the jet is negligible compared to the flux of momentum going into the remainder of the fluid, thereby explaining why it is not necessary to take into account the jet within a momentum analysis. On the other hand, for constant impact velocity, the flux of energy going into the jet is equal to half the work performed by the body. To leading order with respect to the deadrise angle, the asymptotic

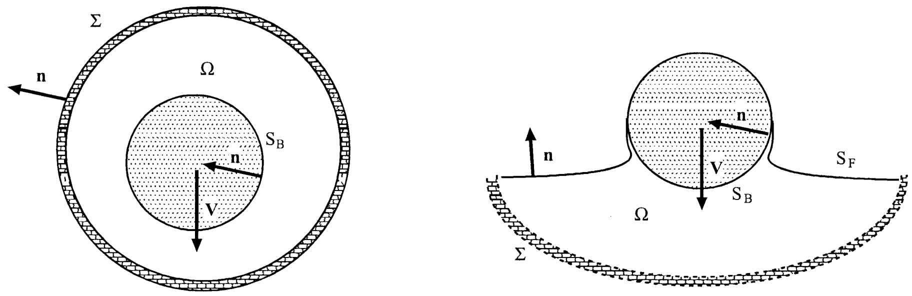

Figure 1. Geometric definitions. Left: solid boundaries. Right: vertical impact case.

solution therefore predicts the equal distribution of energy between the jet and the remainder of the fluid. This property does not depend on the exact shape of the body. This can be checked by analyzing the different asymptotic solutions for cylinders [16], wedges [17], and more generally arbitrary symmetric [18, 19] blunt bodies. Three-dimensional examples, such as axisymmetric shapes [19] or elliptic paraboloid [23] also confirm that the loss of energy within the jet can resolve the contradiction between the two approaches. Finally, examples of computations are given, together with comparisons between asymptotic and experimental results.

2. Momentum and energy conservation

The problem is formulated within the framework of potential-flow theory. The fluid is assumed to be incompressible and inviscid, the flow is irrotational and the body is rigid. In the fluid domain, the velocity potential ϕ satisfies the Laplace equation

$$\Delta\phi = 0, \tag{1}$$

together with the boundary condition

$$\phi_n = V \cdot n, \tag{2}$$

on the rigid boundaries. Here V is the velocity of the boundary and n its normal.

2.1. Solid boundaries

In this section, we consider a body moving inside a fluid domain limited by solid boundaries. An illustrative case is a cylinder moving inside another cylinder, along a diameter (see Figure 1).

2.1.1. *Momentum conservation*

The hydrodynamic force acting on the body is obtained by integration of the dynamic pressure p on S_B, the wetted area of the body, leading to

$$F = -\rho \int\int_{S_B} (\phi_t + \frac{1}{2}(\nabla\phi)^2)\, n\mathrm{d}s. \tag{3}$$

Following Newman [25, pp. 132–134], we can write

$$F = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\int_{S_B} \phi\, n\mathrm{d}s\right) - \rho\int\int_{\Sigma}\left(\phi_n\nabla\phi - \frac{1}{2}(\nabla\phi)^2 n\right)\mathrm{d}s, \tag{4}$$

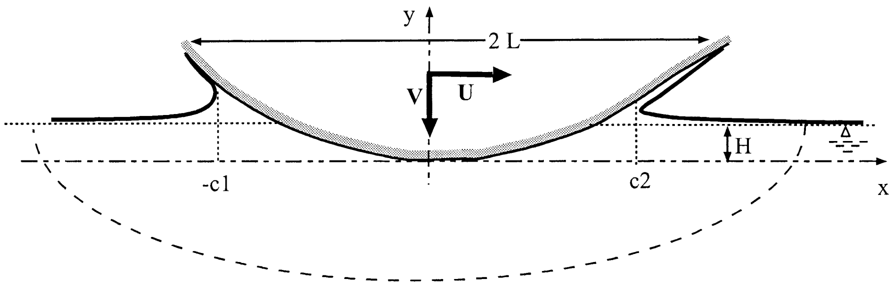

Figure 2. impact problem for small dead-rise angles.

where Σ refers to the rigid boundary surrounding the fluid. Σ being fixed, and by definition of the added-mass M_a, we finally get

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}(M_a V) + \rho \int\int_{\Sigma} \frac{1}{2}(\nabla\phi)^2\, n\mathrm{d}s. \tag{5}$$

In this equation, the last term corresponds to the reaction force acting on Σ, so that momentum is conserved globally.

2.1.2. *Energy arguments*
The kinetic energy in the fluid domain Ω is

$$\begin{aligned} E_k &= \frac{1}{2}\rho \int\int\int_{\Omega} (\nabla\phi)^2\, \mathrm{d}v \qquad (6)\\ &= \frac{1}{2}\rho \int\int_{S_B \cup \Sigma} \phi\phi_n\, \mathrm{d}s = \frac{1}{2}\rho \int\int_{S_B} \phi\phi_n\, \mathrm{d}s \\ &= \frac{1}{2} M_a\, V^2, \qquad (7) \end{aligned}$$

in view of (2). The time derivative of E_k equal to

$$\begin{aligned} \frac{\mathrm{d}E_k}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int\int\int_{\Omega} (\nabla\phi)^2 \mathrm{d}v\right) \\ &= \frac{1}{2}\rho \int\int\int_{\Omega} \frac{\partial}{\partial t}(\nabla\phi)^2 \mathrm{d}v + \frac{1}{2}\rho \int\int_{S_B\cup\Sigma} (\nabla\phi)^2 \phi_n \mathrm{d}s \\ &= \rho \int\int\int_{\Omega} \nabla\phi_t \cdot \nabla\phi \mathrm{d}v + \frac{1}{2}\rho \int\int_{S_B\cup\Sigma} (\nabla\phi)^2 V \cdot n\mathrm{d}s \\ &= \rho \int\int_{S_B\cup\Sigma} \phi_t \nabla\phi \cdot n\mathrm{d}s + \frac{1}{2}\rho \int\int_{S_B\cup\Sigma} (\nabla\phi)^2 V \cdot n\mathrm{d}s \\ &= \rho \int\int_{S_B} (\phi_t + \frac{1}{2}(\nabla\phi)^2) V \cdot n\mathrm{d}s \\ &= -F \cdot V \end{aligned} \tag{8}$$

Hence, in the simple case considered here, energy conservation gives

$$F = \frac{1}{2V}\frac{\mathrm{d}}{\mathrm{d}t}(M_a V^2) = \frac{\mathrm{d}}{\mathrm{d}t}(M_a V) - \frac{1}{2} V \frac{\mathrm{d}M_a}{\mathrm{d}t}, \tag{9}$$

while momentum analysis gives (see (5))

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(M_a V) - \rho \int\int_{\Sigma} \frac{1}{2}(\nabla\phi)^2 n_z \mathrm{d}s. \tag{10}$$

The two expressions for the force (9) and (10) must be equal from physical reasoning. Formally, this equality can also be demonstrated following Wu [26]. For the problem considered here, momentum and energy analysis therefore yield the same results. In passing, it is interesting to note that Equation (9) is usually simpler to use in a numerical computation.

2.2. The impact case

The case of a symmetric body entering calm water at high speed is now considered (see Figure 1). Since the free surface is a material surface at zero pressure, the previous analysis based on momentum conservation remains valid if the pressure is integrated on both the wetted part of the body and the free surface. Formally, S_B can be replaced with $S_B \cup S_F$ in Equations (3) to (4). Taking Σ to infinity gives the slamming force

$$F = \rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\int\int_{S_B \cup S_F} \phi n \mathrm{d}S\right). \tag{11}$$

In water impact problems, the fully nonlinear dynamic free-surface condition:

$$\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi + g\eta = 0 \quad \text{on} \quad z = \eta \tag{12}$$

is classically linearized into the simplified condition $\phi = \phi_t = 0$ on $z = 0$ (or equivalently on S_{F0}), therefore neglecting gravity and nonlinear effects. As will be shown in the next section, this approximation is indeed justified for small time values, far from the body. Within this model, the contribution of the free surface vanishes and the slamming force is given by

$$F = -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\int\int_{S_{B0}} \phi n \mathrm{d}S\right) = \frac{\mathrm{d}}{\mathrm{d}t}(M_a V), \tag{13}$$

where the added mass has to be computed from the simplified problem, *i.e.*, using a Dirichlet free-surface condition on S_{F0} and a Neumann boundary conditon on S_{B0}, the corresponding wetted part of the body.

Simarly, the exact formulation of the kinetic energy

$$E_k = \frac{1}{2}\rho \int\int\int_{\Omega} (\nabla\phi)^2 \, \mathrm{d}v = \frac{1}{2}\rho \int\int_{S_B \cup S_F} \phi\phi_n \mathrm{d}s \tag{14}$$

reduces apparently to

$$E_k = \frac{1}{2}\rho \int\int_{S_{B0}} \phi\phi_n \mathrm{d}s = \frac{1}{2} M_a V^2 \tag{15}$$

so that we obtain as before in the simple case considered

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(M_a V) - \frac{1}{2} V \frac{\mathrm{d}M_a}{\mathrm{d}t}. \tag{16}$$

In view of (13) and (16) there is apparently no hope, this time, to resolve the contradiction. The problem actually results from the fact that the simplified free-surface condition $\phi = \phi_t =$

0 has been used instead of the exact free-surface condition (12). It is well known that the simplified solution, based on the homogeneous Dirichlet free-surface condition, is singular at the intersection between the body shape and the outer expansion of the free surface. Following the work of Wagner [8], detailed studies of the singularity were performed (see *e.g.* [17] and [27]). In particular, the quadratic term must not be neglected in the immediate vicinity of the body. Physically, the spatial features of the flow are changing rapidly in this region where jets developing along the body are observed (see *e.g.* the pictures by Greenhow [5]). In the next section, the situation is highlighted in view of the two-dimensional asymptotic solution. It is shown that the use of the simplified free-surface condition does not allow the flux of energy through the jets to be taken into account.

3. Blunt-body asymmetric impact problem

3.1. Far-field flow (outer solution)

The problem of a blunt body striking the free surface of an inviscid and incompressible fluid is considered (see Figure 2). For the sake of simplicity, the body velocity, $V = U \cdot e_x - V \cdot e_y$ is assumed to be constant. The problem is formulated in a reference frame fixed with respect to the body so that the total velocity potential can be written as

$$\phi = -Ux + Vy + \varphi. \tag{17}$$

The shape of the body is given by $y/h = f(x/L)$ where the length scales H and L correspond to the penetration depth and the half wetted length, respectively. These last two quantities may depend on time. The solution procedure relies on the method of matched asymptotic expansions. The exact solutions for the potential $\tilde{\varphi}$ and the free-surface elevation $\tilde{\eta}$ are expanded as functions of the small perturbation parameter $\epsilon = H/L$ according to

$$\tilde{\varphi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}; \epsilon) = \tilde{\mu}_1(\epsilon)\tilde{\varphi}_1(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) + o(\tilde{\mu}_1), \tag{18}$$

$$\tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t}; \epsilon) = \tilde{\nu}_1(\epsilon)\tilde{\eta}_1(\tilde{x}, \tilde{y}, \tilde{t}) + o(\tilde{\nu}_1), \tag{19}$$

where the following notations are used

$$\tilde{x} = \frac{x}{L},\ \tilde{y} = \frac{y}{L},\ \tilde{t} = \frac{Vt}{H},\ \tilde{\eta} = \frac{\eta}{H},\ \tilde{\varphi} = \frac{\varphi}{VL}. \tag{20}$$

The non-dimensional variables are assumed to be of order $O\,(1)$, since we are seeking a far-field solution with a characteristic length scale of the same order as the body wetted width. The amplitude of the perturbation is given by the gauge functions $\tilde{\mu}_1$ and $\tilde{\nu}_1$ which depend only on the perturbation parameter. Folowing Kevorkian and Cole [28], these functions are to be determined using the principle of least degeneracy.

The choices for the length and velocity scales used to make φ and η non-dimensional are therefore arbitrary. Using the non-dimensional variables (20), we have the following the body boundary condition

$$\epsilon\left(-\frac{U}{V} + \tilde{\mu}_1\tilde{\varphi}_{1\tilde{x}}\right) f_{\tilde{x}} - 1 - \tilde{\mu}_1\tilde{\varphi}_{1\tilde{y}} = 0, \quad \text{on} \quad \tilde{y} = \epsilon f(\tilde{x}). \tag{21}$$

The effects of the horizontal velocity can therefore be neglected when $\epsilon U/V \ll O(1)$. The principle of least degeneracy leads then to impose $\tilde{\mu}_1 = 1$. To leading order, the simplified body boundary condition reduces to

$$\tilde{\varphi}_{1\tilde{y}} = -1 \quad \text{on} \quad \tilde{y} = 0, \quad -\tilde{c}_1 < \tilde{x} < \tilde{c}_2, \tag{22}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ have been introduced to account for the asymmetry of the body. They represent the abscissas of the intersection points between the outer expansion of the free-surface elevation and the body shape (see Figure 2).

The kinematic free-surface condition

$$\tilde{\eta}_{1\tilde{t}} + \epsilon\left(-\frac{U}{V} + \tilde{\varphi}_{1\tilde{x}}\right)\tilde{\eta}_{1\tilde{x}} - \frac{1}{\tilde{\nu}_1}(1 + \tilde{\varphi}_{1\tilde{y}}) = 0 \quad \text{on} \quad \tilde{\eta} = \epsilon\tilde{\nu}_1\tilde{\eta}_1, \tag{23}$$

leads to the conclusion that the order of magnitude of the free-surface elevation is $\tilde{\nu}_1 = 1$. To leading order, this condition reduces to

$$\tilde{\eta}_{1\tilde{t}} = 1 + \tilde{\varphi}_{1\tilde{y}} \quad \text{on} \quad \tilde{y} = 0, \quad \tilde{x} < -\tilde{c}_1, \quad \tilde{x} > \tilde{c}_2. \tag{24}$$

Finally, the dynamic free-surface condition

$$\tilde{\varphi}_{1\tilde{t}} + \epsilon\left(-\frac{U}{V}\tilde{\varphi}_{1\tilde{x}} + \tilde{\varphi}_{1\tilde{y}} + \frac{1}{2}(\tilde{\varphi}_{1\tilde{x}}^2 + \tilde{\varphi}_{1\tilde{y}}^2)\right) + \epsilon\frac{gH}{V^2}\tilde{\eta}_1 = 0 \tag{25}$$

shows that the effects of gravity can be neglected as long as condition $\epsilon gH/V^2 \ll O(1)$ remains satisfied. In that case, it reduces to

$$\tilde{\varphi}_{1\tilde{t}} = 0 \quad \text{on} \quad \tilde{y} = 0, \quad \tilde{x} < -\tilde{c}_1, \quad \tilde{x} > \tilde{c}_2, \tag{26}$$

which can be integrated in time, starting with some initial conditions. For a fluid initially at rest, this leads to the homogeneous Dirichlet condition which is classically applied to the potential on the undisturbed position of the free surface. Within this model, fluid particles on the free surface have only a vertical velocity.

In order to render the analytical resolution easier, the simplified problem is made symmetric by the introduction of the following variables

$$\bar{x} = \tilde{x} + (\tilde{c}_1 - \tilde{c}_2)/2, \quad \bar{y} = \tilde{y}, \quad \bar{c} = (\tilde{c}_1 + \tilde{c}_2)/2, \quad \bar{\varphi} = \tilde{\varphi}_1, \tag{27}$$

so that the harmonic potential $\bar{\varphi}_1$ must satisfy the resulting boundary conditions

$$\bar{\varphi}_{1\bar{y}} = -1 \quad \bar{y} = 0 \quad |\bar{x}| < \bar{c} \tag{28}$$

$$\bar{\varphi}_1 = 0 \quad \bar{y} = 0, \quad |\bar{x}| > \bar{c}. \tag{29}$$

This boundary-value problem also describes the flow around a flat plate moving with a constant velocity into an unbounded fluid. The velocity potential on the body is classically given by

$$\bar{\varphi}_1 = -(\bar{c}^2 - \bar{x}^2)^{\frac{1}{2}}. \tag{30}$$

Although the potential is continuous at the intersections ($|\tilde{x}| = \bar{c}$), the outer solution is nevertheless singular since the vertical velocity of the fluid is unbounded at these points. As will be briefly presented in the next section, the singularities can be removed by matching the outer solution to nonlinear inner solutions describing the formation of jets along the sides of the body.

3.2. SPRAY ROOTS AND JET SOLUTIONS

The spray-root solution describing the flow in the immediate vicinity of the intersection point is presented in detail by Cointe [17] and Howison *et al.* [19]. Although the body shape is asymmetric in the present study, the main results remain unchanged due to the local character of the spray-root solution. For the sake of completeness, the asymptotic analysis near the contact point $\tilde{x} = \tilde{c}_2$ is summarized below. The results are directly transposable to the other extremity ($\tilde{x} = -\tilde{c}_1$).

The following inner variables are introduced to describe the evolution of the flow in the spray-root domain

$$x^\star = (\tilde{x} - \tilde{c}_2)/\epsilon^2, \quad y^\star = (\tilde{y} - \epsilon f(\tilde{c}_2))/\epsilon^2, \quad \epsilon\varphi^\star = \tilde{\varphi}. \tag{31}$$

The space variables are therefore stretched so that the characteristic length scale is ϵH, *i.e.*, one order of magnitude smaller than the penetration depth. The fully nonlinear problem would have been recovered if the choice of H as length scale had been made. The velocity scale V/ϵ is one order of magnitude larger than in the outer domain, in agreement with the intuitive notion that higher fluid velocities are expected near the spray root. This choice also results from applying the principle of least degeneracy to the free-surface boundary conditions. Finally, the change of variable

$$\varphi^\star = \varphi_1^\star - \frac{\mathrm{d}\tilde{c}_2}{\mathrm{d}\tilde{t}} x^\star \tag{32}$$

is made since the spray root is moving along the body with horizontal velocity $\mathrm{d}\tilde{c}_2/\mathrm{d}\tilde{t}$. To leading order with respect to the perturbation parameter, the inner potential satisfies the Laplace equation, subject to the body boundary condition $\varphi^\star_{1y^\star} = 0$ on $y^\star = 0$, and the free-surface conditons on $S(x^\star, y^\star) = 0$

$$\nabla^\star\varphi_1^\star \cdot \nabla^\star S = 0, \tag{33}$$

$$\nabla^\star\varphi_1^\star \cdot \nabla^\star\varphi_1^\star = \left(\frac{\mathrm{d}\tilde{c}_2}{\mathrm{d}t}\right)^2. \tag{34}$$

The kinematic free-surface condition (33) states that the velocity on the free surface is everywhere directed along the tangent to the free-surface. The dynamic free surface condition (34) states that the modulus of the velocity on the free surface is constant. This classical nonlinear boundary-value problem is solved using conformal mapping. The solution exhibits a jet of constant velocity $2\mathrm{d}\tilde{c}_2/\mathrm{d}\tilde{t}$ and constant tickness $\tilde{\delta}_2$ which is determined by matching the inner and outer solutions. Following Cointe [17], the outer expansion of the inner solution near $\tilde{x} = \tilde{c}_2$ is given by

$$\tilde{\varphi} \simeq -4(\delta_2^\star/\pi)^{\frac{1}{2}}(\mathrm{d}\tilde{c}_2/\mathrm{d}\tilde{t})(\tilde{c}_2 - \tilde{x})^{\frac{1}{2}}, \tag{35}$$

while the inner expansion of the outer solution is

$$\tilde{\varphi} \simeq -(\tilde{c}_1 + \tilde{c}_2)^{\frac{1}{2}}\,(\tilde{c}_2 - \tilde{x})^{\frac{1}{2}}, \quad \text{for} \quad \tilde{x} < \tilde{c}_2 \tag{36}$$

according to (30). The matching conditon states that the outer expansion of the inner solution must be equal to the inner expansion of the outer solution, leading to $\delta_2^\star = \tilde{\delta}_2/\epsilon^2 = \pi(\tilde{c}_2 +$

$\tilde{c}_1)/[4\mathrm{d}\tilde{c}_2/\mathrm{d}t]^2$ for the jet tickness. When dimensional variables are used, the thickness of the spray sheet, the fluid velocity within the jet, and the spray-root velocity near $x = c_2$ and $x = -c_1$ are finally given by

$$\delta_2 = \pi V^2(c_2 + c_1)/[4\mathrm{d}c_2/\mathrm{d}t]^2, \quad V_{2j} = 2\frac{\mathrm{d}c_2}{\mathrm{d}t} \quad V_{2\mathrm{sr}} = \frac{\mathrm{d}c_2}{\mathrm{d}t}, \tag{37}$$

$$\delta_1 = \pi V^2(c_2 + c_1)/[4\mathrm{d}c_1/\mathrm{d}t]^2, \quad V_{1j} = 2\frac{\mathrm{d}c_1}{\mathrm{d}t} \quad V_{1\mathrm{sr}} = \frac{\mathrm{d}c_1}{\mathrm{d}t}, \tag{38}$$

According to the present model, the thickness of the jet is constant, which suggests that the length of the jet would be infinite, thus the need for an additional solution would arise. For a symmetric wedge, Cointe [18] completed the asymptotic analysis by introducing a solution describing the flow within the jet along the body. The solution predicts a jet with a constant velocity and a thickness which is linearly decreasing along the wedge. This solution also applies in the asymmetric case.

3.3. IMPAC FORCE

The first-order impact force can now be computed using either momentum or energy considerations. Momentum analysis leads to integrate the pressure on the body wetted surface. The composite solution for the pressure is obtained by adding the inner and outer expansions, and by subtracting the common part of the two expansions. To first order, the impact force is obtained by integration of the dynamic pressure in the outer domain, the contribution from the spray-root domain being of a higher order. Using dimensional variables, we obtain the potential on the body as $\varphi^{(1)} = -V[(c_2 - x)(c_1 + x)]^{\frac{1}{2}}$. Applying (13) we obtain the leading-order term for the impact force as

$$F^{(1)} = \frac{\pi}{4}\rho V(c_1 + c_2)(\dot{c}_1 + \dot{c}_2) = V\frac{\mathrm{d}M_a}{\mathrm{d}t}. \tag{39}$$

For the energy balance, the rate of kinetic energy E_k^{out} transferred from the body to the far field (or outer domain) is first computed. Since $\varphi^{(1)}$ satisfies a Dirichlet condition on the undisturbed free surface, Equation (15) can be used directly, leading to

$$\begin{aligned}\frac{\mathrm{d}E_k^{\mathrm{out}}}{\mathrm{d}t} &= \frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{-c_1}^{c_2} -V\varphi^{(1)}\mathrm{d}x = \frac{1}{2}\rho V\int_{-c_1}^{c_2}\frac{\mathrm{d}}{\mathrm{d}t}[(c_2 - x)(c_1 + x)]^{\frac{1}{2}}\mathrm{d}x \\ &= \frac{\pi}{8}\rho V^2(c_1 + c_2)(\dot{c}_1 + \dot{c}_2) = \frac{1}{2}V^2\frac{\mathrm{d}M_a}{\mathrm{d}t},\end{aligned} \tag{40}$$

so that the following equality holds

$$\frac{\mathrm{d}E_k^{\mathrm{out}}}{\mathrm{d}t} = \frac{1}{2}F^{(1)}\cdot V \tag{41}$$

According to the asymptotic analysis, half the energy transferred from the body to the fluid is therefore imparted to the main flow. The other missing half is to be found within the jets developing near the intersections. Taking into account the spray-root velocities $V_{1\mathrm{sr}} = V_{1j}/2$ and $V_{1\mathrm{sr}} = V_{1j}/2$, the flux of energy G through the two jets is

$$G_j = \delta_1\frac{1}{2}\rho V_{1j}^2(V_{1j} - V_{1\mathrm{sr}}) + \delta_2\frac{1}{2}\rho V_{2j}^2(V_{2j} - V_{2\mathrm{sr}}) \tag{42}$$

$$= \frac{\pi}{8}\rho V^2(c_2+c_1)\dot{c}_1 + \frac{\pi}{8}\rho V^2(c_2+c_1)\dot{c}_2 = \frac{\pi}{8}\rho V^2(c_2+c_1)(\dot{c}_1+\dot{c}_2)$$
$$= \frac{1}{2}V^2\frac{\mathrm{d}M_a}{\mathrm{d}t} = \frac{\mathrm{d}E_k^{\mathrm{out}}}{\mathrm{d}t}. \tag{43}$$

Equation (43) shows that as much kinetic energy has been imparted to the jets as to the remainder of the fluid. At this stage, the relation between the wetted length and the body shape has not yet been introduced. The energy distribution between the jets and the main flow does not depend on the exact shape of the body but would appear as an intrinsic property of the flow. To leading order, the energy balance finally yields

$$F^{(1)} \cdot V = \frac{\mathrm{d}E_k^{\mathrm{out}}}{\mathrm{d}t} + G_j. \tag{44}$$

This relation can be checked by analyzing the different asymptotic solutions for cylinders [16], [17] and more generally arbitrary summetric ([18], [19]) blunt bodies. Three-dimensional examples, such as axisymmetric shapes [19] or elliptic paraboloid [23], [24] confirm the generality of this result for constant vertical velocity.

In view of the asymptotic distribution of energy, the general results obtained in Section 2.2 with the homogeneous Dirichlet free-surface condition for the potential can now be re-interpreted. In the case of a constant velocity, Equations (13) and (16) give two expressions for the force which differ by a factor of two. Since the flux of energy has not been taken into account in the energy analysis, this suggests that exactly half of the work done by the body has been transformed into kinetic energy within the jets, independently of the body shape (blunt or not). This result, which applies in the so-called inertia stage, assumes that compressibility and gravity effects are negligible. It is also interesting to consider the analogy between the impact and planning problems. Within the framework of the nonlinear high-Froude-number slender-body theory [29], the high-speed planning problem for a slender ship and the two dimensional impact problem are identical to leading order.

4. Examples of computation

In this section, the wetted corrections c_1 and c_2 are first determined for arbitrary two-dimensional body shapes. Examples of asymptotic computations are then presented, together with comparisons between asymptotic and experimental results.

4.1. Evaluation of the wetted length

The composite asymptotic solution described in the previous section depends on the wetted lengths c_1, c_2. Unlike energy repartition, these quantities are strongly dependent on the body shape and remain unknown at this stage, hence the need for additional closure equations. Although Laplace's equation is satisfied, the asymptotic solution does not automatically satisfy volume conservation as a result of the singularity arising at the initial time. Volume conservation has to be imposed explicitly as a closure equation (see *e.g.* [19], [30]). To leading order, volume conservation is satisfied when the outer expansion of the free surface intersects the exact body shape. The resulting closure equations are nevertheless complicated to solve, especially in the case of an asymmetric body shape. In the present study, the wetted length is computed using the displacement potential

$$\tilde{\Phi}_1(\tilde{x},\tilde{y},\tilde{t}) = \int_0^{\tilde{t}} \tilde{\varphi}_1(\tilde{x},\tilde{y},\tau)\mathrm{d}\tau, \tag{45}$$

as suggested by Korobkin [20]. The displacement potential is harmonic in the fluid domain and satisfies along the $\bar{x}$-axis the mixed boundary conditions, $\tilde{\Phi}_{1\tilde{y}} = f(\tilde{x}) - \tilde{t}$ on the body ($\bar{y} = 0$, $|\bar{x}| < \bar{c}$), and $\tilde{\Phi}_{1\tilde{x}} = 0$ on the free surface ($\bar{y} = 0$, $|\bar{x}| > \bar{c}$). Following Newman ([25], pp. 180–184), a formal solution can be obtained by transforming this 'lifting' problem into an equivalent 'thickness' problem for the normal derivative of the displacement potential. It is found that this last quantity satisfies the integral equation

$$\tilde{\Phi}_{1\tilde{y}}(\tilde{x}, \tilde{y}, \tilde{t}) = \frac{1}{\pi[(\tilde{c}_1 + \tilde{x})(\tilde{c}_2 - \tilde{x})]^{\frac{1}{2}}} \int_{-\tilde{c}_1}^{\tilde{c}_2} \tilde{\Phi}_{1\tilde{y}}(s, 0, \tilde{t}) \frac{[(\tilde{c}_1 + s)(\tilde{c}_2 - s)]^{\frac{1}{2}}}{s - \tilde{x}} \, \mathrm{d}s, \tag{46}$$

corresponding to a distribution of sources located along the body. In view of (24), the normal derivative of the displacement potential on the free surface is also equal to the free-surface vertical displacement in a frame of reference fixed with respect to the fluid. For the outer expansion of the free-surface elevation to be finite at the intersections ($|\bar{x}| = \bar{c}$), the integral in Equation (46) must vanish at these points, leading to

$$\int_{-\tilde{c}_1}^{\tilde{c}_2} f(\tilde{x}) \left(\frac{\tilde{c}_2 - \tilde{x}}{\tilde{c}_1 + \tilde{x}}\right)^{\frac{1}{2}} \mathrm{d}\tilde{x} = h(\tilde{t}) \int_{-\tilde{c}_1}^{\tilde{c}_2} \left(\frac{\tilde{c}_2 - \tilde{x}}{\tilde{c}_1 + \tilde{x}}\right)^{\frac{1}{2}} \mathrm{d}\tilde{x}, \tag{47}$$

$$\int_{-\tilde{c}_1}^{\tilde{c}_2} f(\tilde{x}) \left(\frac{\tilde{c}_1 + \tilde{x}}{\tilde{c}_2 - \tilde{x}}\right)^{\frac{1}{2}} \mathrm{d}\tilde{x} = h(\tilde{t}) \int_{-\tilde{c}_1}^{\tilde{c}_2} \left(\frac{\tilde{c}_1 + \tilde{x}}{\tilde{c}_2 - \tilde{x}}\right)^{\frac{1}{2}} \mathrm{d}\tilde{x}, \tag{48}$$

where *h(t)* is the immersion depth. When $\tilde{c}_1 = \tilde{c}_2$, the combination of these last two equations leads back to the classical equation of the symmetric case, which is a well-known integral equation [31, p. 229]. A quasi-analytical solution of these equations for arbitrary polynomial body shapes is given in the appendix. The particular results derived by Toyama [21] for polynomial body shapes up to the power four have been recovered and extended. In the case of arbitrary body shapes, it is also described in the appendix how the equations for the wetted lengths can be solved accurately by a numerical procedure. Finally, once the wetted lengths $\tilde{c}_1$ and $\tilde{c}_2$ have been obtained, the free-surface elevation can be computed according to

$$\tilde{\eta}_1 = \frac{1}{\pi} \frac{1}{\sqrt{(\tilde{x} + \tilde{c}_1)(\tilde{x} - \tilde{c}_2)}} \int_{-\tilde{c}_1}^{\tilde{c}_2} \tilde{\Phi}_{1\tilde{y}}(\xi) \frac{\sqrt{(\tilde{c}_1 + \xi)(\tilde{c}_2 - \xi)}}{\tilde{x} - \xi} \, \mathrm{d}\xi, \tag{49}$$

where the Neumann condition on the body is: $\tilde{\Phi}_{1\tilde{y}}(\xi) = f(\xi) - \tilde{t}$.

4.2. Examples of computations

The asymptotic solution can be used to describe several impact situations. As soon as the geometry near the contact point is sufficiently flat, the leading-order boundary-value problems are identical whether solid-fluid or fluid-fluid impact problems are considered. The case of an asymmetric body with a parabolic shape striking the free surface of fluid initially at rest is presented in Figure 3. The impact of a fluid drop or jet tip on a planar rigid surface is illustrated in Figure 4. Such an application was initially considered by Cumberbatch [32] who studied the impact of a fluid wedge onto a flat wall. Only the far-field free-surface elevations have been represented, but the complete composite solution would exhibit jets with decreasing thickness along the wall. The case of an asymmetric body striking an asymmetric free surface is presented in Figure 5.

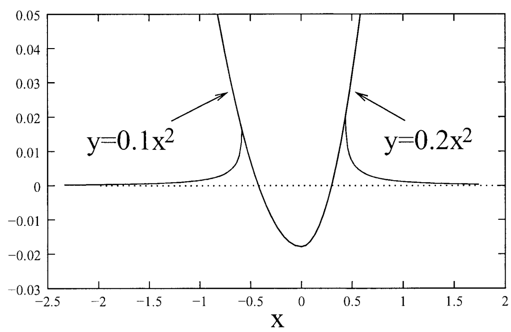

Figure 3. Free-surface deformation due to the penetration at constant vertical velocity of a parabolic asymmetric blunt body into a free surface initially at rest.

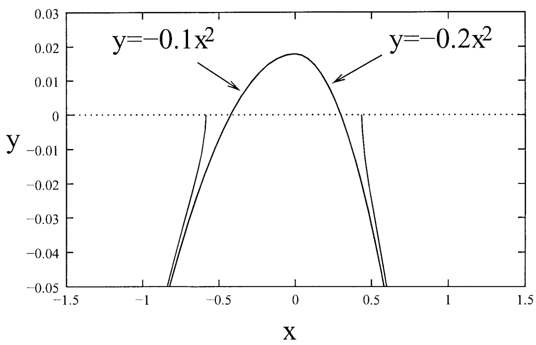

Figure 4. Free-surface deformation of a parabolic asymmetric jet tip or droplet impacting a rigid wall with constant velocity.

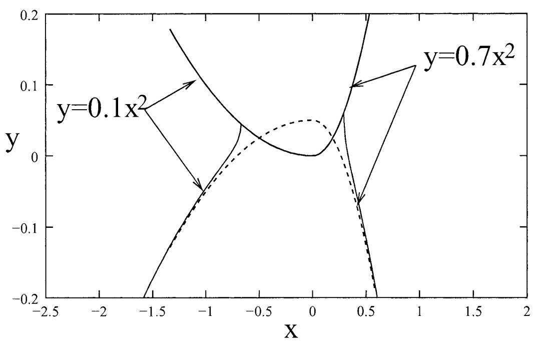

Figure 5. Free-surface deformation of two asymmetric droplets or jet tips with similar parabolic shapes striking each other with constant vertical velocities.

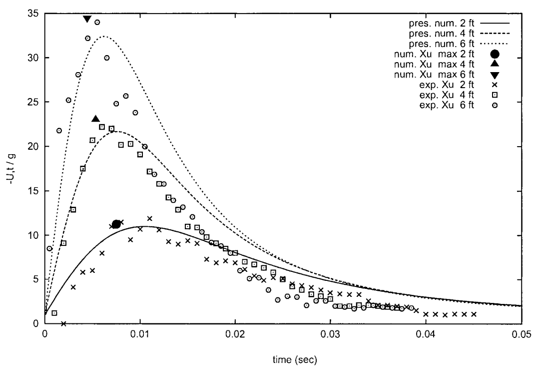

Figure 6. Comparison between numerical and experimental results for the asymmetric free-fall drop test of a wedge.

Finally, the experiments by Xu *et al.* [33] for the free fall of a wedge onto an initially undisturbed free surface have been reproduced numerically. The time-domain simulation of the impact has been performed using the loads derived by integration of the composite solution proposed by Zhao and Faltinsen [34].

The wedge acceleration is plotted as a function of time for different drop heights in Figure 6. The agreement between asymptotic and experimental results is reasonable, especially for the shortest times. The maximum acceleration and its time of occurrence are correctly predicted. Nevertheless, the asymptotic solution tends to overpredict the acceleration in the decaying phase.

5. Conclusion

It is shown that the impact load can be evaluated using the concept of added mass through either a momentum or an energy analysis without any contradiction. In order to derive the energy approach, it is nevertheless necessary to take into account the flux of energy going through the jets. The energy distribution in the inertia stage is also studied. For a blunt body, the asymptotic theory predicts to leading order that the energy is equally transferred to the jets and the remainder of the fluid, independently of body shape. The present study suggests that this property remains satisfied in the the case of arbitrary body shapes.

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Appendix: A. Evaluation of the wetted lengths

A.I. Analytical results for polynomial shapes

When the function *f(x)* describing the body shape is a polynomial function, a quasi-analytical solution of Equations (47) and (48) for the wetted lengths can be found. In order to take into account a possible discontinuity of the slope at the lower point (chosen as the origin), two polynomials are introduced on both sides with respect to the origin as follows:

$$f(x) = \begin{cases} \sum_{i=1}^{\infty} a_{1i}|x|^i \text{ for } x \leq 0 \\ \sum_{i=1}^{\infty} a_{2i}|x|^i \text{ for } x > 0. \end{cases} \tag{A1}$$

Substituting *f(x)* in the integrals from $-c_1$ to c_2 gives

$$\int_{-c_1}^{c_2} f(x)\left(\frac{c_2-x}{c_1+x}\right)^{\frac{1}{2}} \mathrm{d}x = \sum_{i=1}^{\infty} a_{1i}(-1)^i {c_1}^{i+1}\left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} B(1/2, i+1)F(-1/2, i+1, i+3/2; -c_1/c_2) + \sum_{i=1}^{\infty} a_{2i}{c_2}^{i+1}\left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} B(3/2, i+1)F(1/2, i+1, i+5/2; -c_2/c_1) \tag{A2}$$

and

$$\int_{-c_1}^{c_2} f(x)\left(\frac{c_1+x}{c_2-x}\right)^{\frac{1}{2}} \mathrm{d}x = \sum_{i=1}^{\infty} a_{1i}(-1)^i {c_1}^{i+1}\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}} B(3/2, i+1)F(1/2, i+1, i+5/2; -c_1/c_2) + \sum_{i=1}^{\infty} a_{2i}{c_2}^{i+1}\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}} B(1/2, i+1)F(-1/2, i+1, i+3/2; -c_2/c_1), \tag{A3}$$

where B and F denote the Psi and hyper-geometric functions as defined by Equations 6.2.2 and 15.1.1 in [35, pp. 258, 556].

To illustrate and to validate the solution procedure, the case of a symmetric wedge with $a_{11} = -1$ and $a_{21} = 1$ is considered. Introduction of (A2) and (A3) in (47) and (48) yields the two following equations:

$$\frac{\pi}{2}Ut(c_2+c_1) = -a_{11}{c_1}^2\left(\frac{c_1+c_2}{c_1}\right)^{\frac{1}{2}} B(1/2, 2)F(-1/2, 1/2, 5/2; \frac{c_1}{c_1+c_2} + a_{21}{c_2}^2(\left(\frac{c_2}{c_1+c_2}\right)^{\frac{1}{2}} B(3/2, 2)F(1/2, 3/2, 7/2; \frac{c_2}{c_1+c_2}) \tag{A4}$$

and

$$\frac{\pi}{2}Ut(c_2+c_1) = -a_{11}{c_1}^2\left(\frac{c_1}{c_1+c_2}\right)^{\frac{1}{2}} B(3/2,2)F(1/2,3/2,7/2;\ \frac{c_1}{c_1+c_2}$$
$$+a_{21}{c_2}^2(\left(\frac{c_1+c_2}{c_2}\right)^{\frac{1}{2}} B(1/2,2)F(-1/2,1/2,5/2;\ \frac{c_2}{c_1+c_2}), \tag{A4}$$

where the hypergeometric functions are computed from recurrence formulae. In this case, it can be easily shown that c_1 and c_2 vary linearly with time t, as predicted by the classical solution. Nevertheless, the evaluation of the B and F functions requires series to be computed numerically. A more general computational method to estimate the wetted length is presented below.

A.1.1. *Numerical results*

After some algebra, Equations (47) and (48) provide the following two conditions:

$$\int_{-c_1}^{c_2} f(s)\left(\frac{c_2-s}{c_1+s}\right)^{\frac{1}{2}}\,\mathrm{d}s = \int_{-c_1}^{c_2} f(s)\left(\frac{c_1+s}{c_2-s}\right)^{\frac{1}{2}}\,\mathrm{d}s = \frac{\pi}{2}h(t)(c_2+c_1), \tag{A5}$$

where *h(t)* is the draft at instant t. Using the new variables:

$$S = c_2 + c_1, \quad D = c_2 - c_1 \tag{A6}$$

and assuming constant vertical velocity, we can rewrite these two equations in the form

$$\int_0^1 f\left(uS + \frac{D-S}{2}\right)\left(\frac{1-u}{u}\right)^{\frac{1}{2}}\,\mathrm{d}u - \frac{\pi}{2}Vt = 0, \tag{A7}$$

$$\int_0^1 f\left(uS + \frac{D-S}{2}\right)\left(\frac{1-u}{u}\right)^{\frac{1}{2}}\,\mathrm{d}u - \frac{\pi}{2}Vt = 0. \tag{A8}$$

These integrals are solved with a collocation method and the discretization follows from the Mid-Point Rule, leading, for example, to

$$\int_0^1 f\left(uS + \frac{D-S}{2}\right)\left(\frac{1-u}{u}\right)^{\frac{1}{2}}\,\mathrm{d}u \approx \sum_{j=1}^{N} f\left(u_{j+1/2}S + \frac{D-S}{2}\right)$$
$$\int_{u_j}^{u_j+1}\left(\frac{1-u}{u}\right)^{\frac{1}{2}}\,\mathrm{d}u, \tag{A9}$$

where the integrals are now calculated analytically.

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